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d-Lucky Labeling of Graphs

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Abstract

Let $l : V(G) \rightarrow N$ be a labeling of the vertices of a graph G by positive integers. Define $c(u) = \sum_{v\in N(v)} l(v) + d(u)$, where $d(u)$ denotes the degree of u and $N(u)$ denotes the open neighborhood of u. In this paper we introduce a new labeling called d-lucky labeling and study the same as a vertex coloring problem. We define a labeling l as d-lucky if $c(u) \neq c(v)$, for every pair of adjacent vertices u and v in G. The d-lucky number of a graph G, denoted by $\eta_{dl}(G)$, is the least positive k such that G has a d-lucky labeling with $\{1,2,\ldots,k\}$ as the set of labels. We obtain $\eta_{dl}(G) = 2$ for hypercube network, butterfly network, benes network, mesh network, hypertree and X-tree.

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Keywords: Lucky labeling; *d*-lucky labeling; benes network; butterfly network; mesh network.

1. Introduction

Graph coloring is one of the most studied subjects in graph theory. It is an assignment of labels called colors to the elements of a graph, subject to certain constraints. Karonski, Luczak and Thomason[2] initiated the study of proper labeling. The rule of using colors originates from coloring the countries of a map, where each face is colored exactly. In its simplest outline, vertex coloring or proper labeling is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color. The problem of proper labeling offers numerous variants and established great significance at recent times, for example see[1,2,6]. Graph coloring is used in various research areas of computer science such as networking, image segmentation, clustering, image capturing and data mining.

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There is a spectrum of labeling procedures that are available in the literature, leading to proper vertex coloring of graphs. For a mapping $f : V(G) \rightarrow \{1,2,\ldots,k\}$, a proper vertex coloring is obtained through Lucky labeling[9,12], Vertex-labeling by product[5], Vertex-labeling by gap, Vertex-labeling by degree and Vertex-labeling by maximum[5]. For a mapping $f : E(G) \rightarrow \{1,2,\ldots,k\}$, a proper vertex coloring is obtained through Edge-labeling by sum[11], Edge labeling by product[5] and Edge-labeling by gap[5]. In this paper we introduce a new labeling called *d*-lucky labeling and compute the *d*-lucky number of certain networks.

2. Some special classes of graphs with $\eta_{dl}(G) = 2$

We begin with the definition of *d*-lucky labeling. For a vertex u in a graph G, let $N(u) = \{v \in V(G) \,/\, uv \in E(G)\}$ and $N[u] = N(u) \cup \{u\}$.

Definition 2.1 *Let $l : V(G) \rightarrow \{1,2,\ldots,k\}$ be a labeling of the vertices of a graph* G *by positive integers. Define $c(u) = \sum_{v\in N(u)} l(v) + d(u)$, where d(u) denotes the degree of u. We define a labeling l as d-lucky if $c(u) \neq c(v)$, for every pair of adjacent vertices u and v in* G. *The d-lucky number of a grap* G, *denoted by $\eta_{dl}(G)$, is the least positive k such that* G *has a d-lucky labeling with $\{1,2,\ldots,k\}$ as the set of labels.*

Definition 2.2 *The vertex set V of Q_n consists of all binary sequence of length n on the set $\{0,1\}$, that is, $V = \{x_1x_2 \ldots x_n \in \{0,1\}, i = 1,2,\ldots,n\}$. Two vertices are linked by an edge if and only if x and y differ exactly in one coordinate, that is, $\sum_{i=1}^{n} | x_i - y_i | = 1$. In terms of cartesian product, Q_n is defined recursively as follows. $Q_1 = K_2$, $Q_n = Q_{n-1} \times Q_1 = K_2 \times K_2 \times \ldots \times K_2$, $n \geq 2$.*

Theorem 2.3 The *n*-dimensional hypercube network Q_n admits *d*-lucky labeling and $\eta_{dl}(Q_n) = 2$.

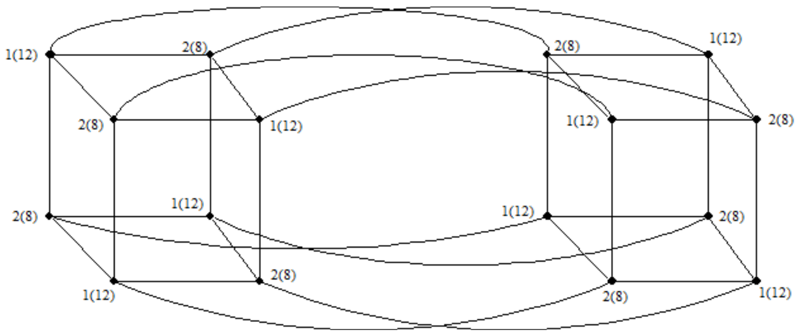

Fig. 1. d -lucky labeling of hypercube network, Q_4 .

A most popular bounded - degree derivative network of the hypercubes is called a butterfly network.

Definition 2.4 [10] *The n-dimensional butterfly network, denoted by BF(n), has a vertex set $V = \{(x;i) : x \in V(Q_n), 0 \leq i \leq n\}$. Two vertices $(x;i)$ and $(y;j)$ are linked by an edge in BF(n) if and only if $j = i+1$ and either*

(*i*) $x = y$, *or*
(*ii*) x *differs from* y *in precisely the* j^{th} *bit.*
For $x = y$, *the edge is said to be a straight edge. Otherwise, the edge is a cross edge. For fixed* i, *the vertex*

$(x;i)$ *is a vertex on level* i.

Definition 2.5 [10] *The topological structure of a mesh network is defined as the Cartesian product* $P_l \times P_m$ *denoted by* $M(l,m)$, *where* P_l *and* P_m *denotes and undirected path on l and m vertices respectively.*

Remark 2.6 *The mesh* $M(l,m)$ *has* lm *vertices and* $2(lm)-(l+m)$ *edges, where* $l,m \geq 2$ *and* l,m *denotes rows and columns of* $M(l,m)$ *respectively.*

Theorem 2.7 The n-dimensional butterfly network BF(n) admits d-lucky labeling and $\eta_{dl}(BF(n)) = 2$.

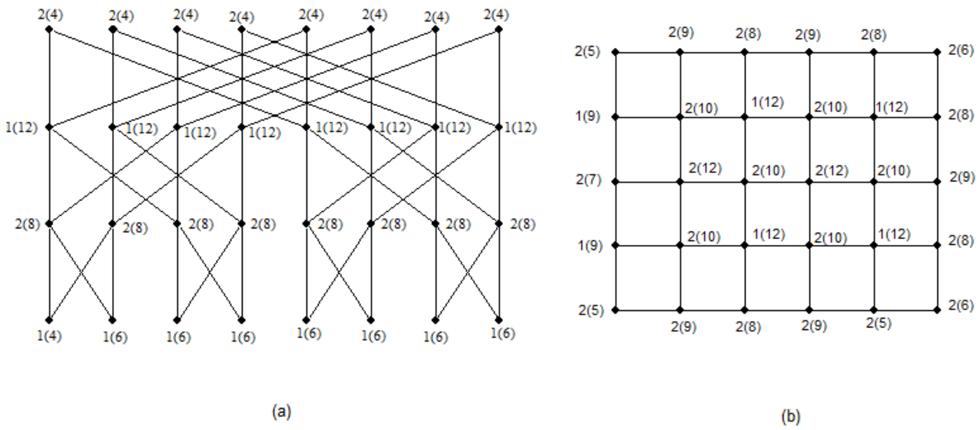

Fig. 2. (a) d-lucky labeling of BF(3); (b) d-lucky labeling of $M(5,6)$ Mesh network.

Proof. Label the vertices in consecutive levels of $BF(n)$ as 1 and 2 alternately, beginning from level 0. We note that every edge $e=(u,v)$ in $BF(n)$ has one end at level i and the other end at level $i+1$ or level $i-1$ (if it exists), $0 \leq i \leq n$.

Case 1: Suppose u is in level 0, then u is incident on one cross edge and one straight edge with the other ends at level 1. Since $l(u)=1$ and each member of $N(u)$ is labeled 2, we have $c(u) = \sum_{v \in N(u)} l(v) + d(u) = 6$, where $d(u)$ is the degree of u. Since $l(v)=2$ and each member of $N(v)$ is labeled 1, we have $c(v) = \sum_{u \in N(v)} l(u) + d(v) = 8$. Thus $c(u) \neq c(v)$. The same argument holds good when u is in level n.

Case 2: Suppose u is in level i, i is even, $0 < i < n$ and v is in level $i+1$. Then u is incident on one cross edge and one straight edge with the other ends at level $i+1$ and also incident on one cross edge and one straight edge with the other ends at level $i-1$. Since $l(u)=2$, each member of $N(u)$ is labeled 1. Therefore, we have $c(u) = \sum_{v \in N(u)} l(v) + d(u) = 8$. Further $l(v)=1$ and each member of $N(v)$ is labeled 2. Therefore, we have $c(v) = \sum_{u \in N(v)} l(u) + d(v) = 12$. Thus $c(u) \neq c(v)$. A similar argument shows that $c(u) \neq c(v)$ if v is in level $i-1$. The case when i is odd is also similar. See Figure 2(a) for d-lucky labeling of $BF(3)$ with *c(u)* listed within paranthesis for any $u \in V$. Hence n-dimensional butterfly network admits d-lucky labeling.

Theorem 2.8 The mesh network denoted by $M(l,m)$ admits d-lucky labeling and $\eta_{dl}(M(l,m)) = 2$..

Proof. Let G be a mesh $M(l,m)$, where $l,m \geq 2$. Then G admits d-lucky labeling and $\eta_{dl}(G) = 2$. Label the vertices in row i, i even, as 1 and 2 alternately, beginning with label 1 from left to right. Label all the vertices in row i, i odd, as 2. Edges with both ends in the same row are called horizontal edges. Edges with one end in row i

and the other end in row $(i+1)$ or row $(i-1)$ are called vertical edges.

Case 1: Suppose u and v are in row 1, where $d(u) = 2$, then u has one horizontal edge and one vertical edge incident at it. If $l(u) = 2$, by labeling of G the adjacent vertices on the horizontal and vertical edges incident with u are labeled as 2 and 1 respectively. We have $c(u) = \sum_{v\in N(u)} l(v) + d(u) = 5$. On the other hand, if v and each member of $N(v)$ is labeled 2, we have $c(v) = \sum_{u\in N(v)} l(u) + d(v) = 9$. Thus $c(u) \neq c(v)$. A similar argument holds when $l(u) = 1$ or when u is in row n.

Case 2: Suppose u and v are in row i, i even, where $d(u) = 3$ and $d(v) = 4$, u has two vertical edges in rows $i-1$ and $i+1$ and one horizontal edge with the other end in row i incident with it. Since $l(u) = 1$, each member of $N(u)$ is labeled 2. Therefore, we have $c(u) = \sum_{v\in N(u)} l(v) + d(u) = 9$. On the other hand, suppose v is in row i, then v has two vertical edges in row $i-1$ and $i+1$ and two horizontal edges with the other end in row i incident with it. Since $l(v) = 2$, each member of $N(v)$ in the horizontal row is labeled 1 and $N(v)$ in vertical column is labeled 2. Therefore, we have $c(v) = \sum_{u\in N(v)} l(u) + d(v) = 10$. The vertex sums are distinct. A similar argument holds when v is in row $n-1$.

Case 3: Suppose u and v are in row i, i odd, where $d(u) = 3$ and $d(v) = 4$, u has two vertical edges in rows $i-1$ and $i+1$ and one horizontal edge with the other end in row i incident with it. Since $l(u) = 2$, by labeling of G the adjacent vertices on the horizontal and vertical edges incident with u are labeled as 2 and 1 respectively. Therefore, we have $c(u) = \sum_{v\in N(u)} l(v) + d(u) = 7$. On the other hand, suppose v is in row i, then v has two vertical edges in row $i-1$ and $i+1$ and two horizontal edges with the other end in row i incident with it. Since $l(v) = 2$, each member of $N(v)$ is labeled 2. Therefore, we have $c(v) = \sum_{u\in N(v)} l(u) + d(v) = 12$. The vertex sums are distinct. (For illustration, see Figure $2(b)$, d-lucky labeling of $M(5,6)$ mesh network with *c(u)* listed within paranthesis for any $u \in V$). Hence the mesh network admits d-lucky labeling with $\eta_{dl}(G) = 2$.

Definition 2.9 *The n-dimensional benes network consists of back-to-back butterfly, denoted by $BB(n)$. The $BB(n)$ has $2n+1$ levels, each with 2^n vertices. The first and last $n+1$ levels in the $BB(n)$ form two $BF(n)'s$ respectively, while the middle level in $BB(n)$ is shared by these butterfly networks. The n-dimensional benes network has $(n+1)2^{n+1}$ vertices and $n2^{n+2}$ edges. It has only 2-degree vertices and 4 -degree vertices, and thus, is eulerian.*

Theorem 2.10 The n -dimensional benes network $\mathrm{BB(n)}$ admits d -lucky labeling and $\eta_{dl}(\mathrm{BB(n)}) = 2$.

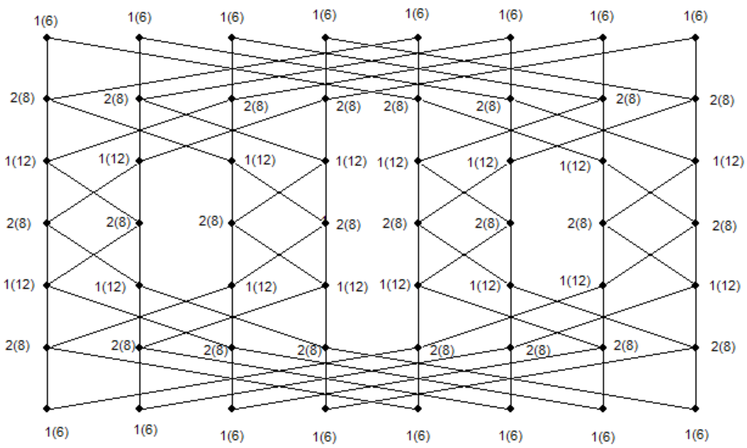

Fig. 3. d -lucky labeling of BB(3).

Proof. Label the vertices in consecutive levels of $BB(n)$ as 1 and 2 alternately, beginning from level 0. We note that every edge $e = (u, v)$ in $BB(n)$ has one end at level i and the other end at level $i+1$ or level $i-1$ (if it exists), $0 \leq i \leq n$.

Case 1: Suppose u is in level 0, then u is incident on one cross edge and one straight edge with the other ends at level 1. Since $l(u) = 2$ and each member of $N(u)$ is labeled 1, we have $c(u) = \sum_{v\in N(u)} l(v) + d(u) = 4$, where $d(u)$ is the degree of u. Since $l(v) = 1$ and each member of $N(v)$ is labeled 2, we have $c(v) = \sum_{u\in N(v)} l(u) + d(v) = 12$. Thus $c(u) \neq c(v)$. The same argument holds good when u is in level n.

Case 2: Suppose u is in level i, i is even, $0 < i < n$ and v is in level $i + 1$. Then u is incident on one cross edge and one straight edge with the other ends at level $i + 1$ and also incident on one cross edge and one straight edge with the other ends at level $i - 1$. Since $l(u) = 2$, each member of $N(u)$ is labeled 1. Therefore, we have $c(u) = \sum_{v\in N(u)} l(v) + d(u) = 8$. Further $l(v) = 1$ and each member of $N(v)$ is labeled 2. Therefore, we have $c(v) = \sum_{u\in N(v)} l(u) + d(v) = 12$. Thus $c(u) \neq c(v)$. A similar argument shows that $c(u) \neq c(v)$ if v is in level i-1. The case when i is odd is also similar. See Figure 3 for d-lucky labeling of $BB(3)$ with $c(u)$ listed within paranthesis for any . Hence n-dimensional benes network admits d-lucky labeling.

Definition 2.11 [13] *A hypertree is an interconnection topology for incrementally expansible multicomputer systems, which combines the easy expandability of tree structures with the compactness of the hypercube; that is, it combines the best features of the binary tree and the hypercube. The basic skeleton of a hypertree is a complete binary tree T_r. Here the nodes of the tree are numbered as follows: The root node has label* 1. *The root is said to be at level* 0. *Labels of left and right children are formed by appending* 0 *and* 1, *respectively to the labels of the parent node. Here the children of the nodes x are labeled as* $2x$ *and* $2x + 1$. *Additional links in a hypertree are horizontal and two nodes in the same level of the tree are joined if their label difference is* 2^{i-2}. *We denote an r-level hypertree as $HT(r)$. It has* $2^{r+1} - 1$ *vertices and* $3(2^r - 1)$ *edges.*

Definition 2.12 *An X-tree XT_n is obtained from complete binary tree on* $2^{n+1} - 1$ *vertices of length* $2^i - 1$, *and adding paths P_i left to right through all the vertices at level i;* $1 \leq i \leq n$.

Theorem 2.13 The r-level hypertree HT_r admits d-lucky labeling and $\eta_{dl}(HT(r)) = 2$.

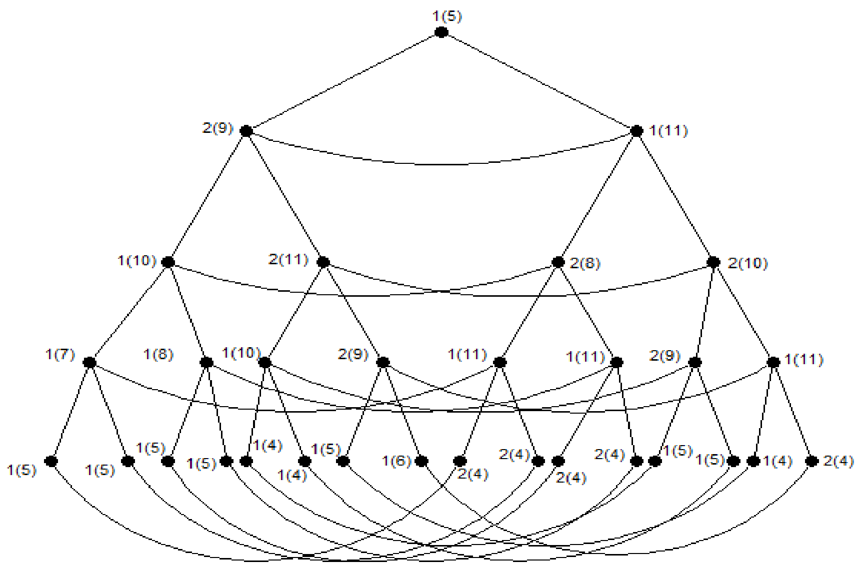

Fig.4. d-lucky labeling of hypertree, $HT(3)$.

Theorem 2.14 The X-tree XT_r admits d-lucky labeling and $\eta_{dl}(XT_r) = 2$

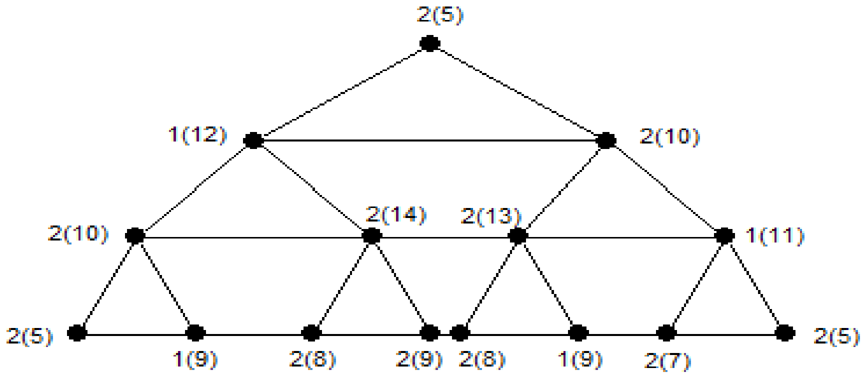

Fig. 5. d -lucky labeling of X -tree, $XT(3)$.

3. Conclusion

A new labeling called *d*-lucky labeling is defined and the graph which satisfies the *d*-lucky labeling is called a *d*-lucky graph. *d*-lucky labeling of some special classes of graphs like hypercube networks, butterfly networks, benes network, mesh network, hypertree and X-tree are investigated.

References

1. S. Czerwinski, J. Grytczuk, V. Zelazny, *Lucky labelings of graphs*, Information Processing Letters, 109(18), 1078-1081 , 2009.
2. M. Karonski, T. Luczak, A. Thomason, *Edge weights and vertex colours*, Journal of Combinatorial Theory,Series B, 91(1), 151-157, 2004.
3. D. Marx, *Graph coloring problems and their applications in scheduling*, Periodica Polytechnica, Mechanical Engineering, Eng. 48(1), 11-16, 2004.
4. S.G. Shrinivas et. al, *Applications of graph theory in computer science an overview*, International Journal of Engineering Science and Technology. 2(9), 4610-4621, 2010.
5. A. Dehghan, M.R. Sadeghi, A. Ahadi, *Algorithmic complexity of proper labeling problems*, Theoretical Computer Science, 495, 25-36, 2013.
6. G. Chartrand, F. Okamoto, P. Zhang, *The sigma chromatic number of a graph*, Graphs and Combinatorics, 26(6), 755- 773, 2010.
7. M.A. Tahraoui, E. Duchene, H. Kheddouci, *Gap vertex-distinguishing edge colorings of graphs*, Discrete Math. 312 (20), 3011-3025, 2012.
8. J.S.-Kaziow, *Multiplicative vertex- colouring weightings of graphs*, Information Processing Letters, 112(5), 191-194, 2012.
9. A. Ahadi, A. Dehghan, M. Kazemi, E. Mollaahmadi, *Computation of lucky number of planar graphs is NP-hard*, Inform. Process. Lett. 112(4), 109-112, 2012.
10. J. Xu, *Topological structure and analysis of interconnection Networks*, Kluwer Academic Publishers,Boston.
11. A. Dudek and D. Wajc, *On the complexity of vertex-coloring edge-weightings*, Discrete Mathematics and Theoretical Computer Science, 13(3), 45-50, 2011.
12. A. Deghan, M.-R. Sadeghia, A. Ahadi, *The complexity of the sigma chromatic number of cubic graphs*, Discrete Appl. Math., submitted.
13. F.F. Dragan, A. Brandstadt, *r-Dominating cliques in graphs with hypertree structure*, Discrete Mathematics, 162, 93-108, 1996.